JOURNAL OF APPROXIMATION THEORY **91**, 226–243 (1997)
ARTICLE NO. AT963095

Asymptotic Behavior of a Poincaré Recurrence System*

Mihály Pituk

Department of Mathematics and Computing, University of Veszprém, P.O. Box 158, 8201 Veszprém, Hungary

Communicated by Paul Nevai

Received May 22, 1996; accepted in revised form October 15, 1996

We establish asymptotic formulae for the solutions of the first order recurrence system $\mathbf{x}_{n+1} = (\mathbf{A} + \mathbf{B}_n)\,\mathbf{x}_n$, where $\mathbf{A}$ and $\mathbf{B}_n$ $(n = 0, 1, ...)$ are square matrices and $\sum_{n=0}^{\infty} \|\mathbf{B}_n\|^2 < \infty$. As a consequence, we confirm a recent conjecture about the asymptotic behavior of the solutions of the higher order scalar equation $u(n+1) = \sum_{i=0}^{k} (c_i + d_i(n))\, u(n-i)$.

1. INTRODUCTION AND MAIN RESULTS

Consider the system of first order recurrence equations

$$\mathbf{x}_{n+1} = (\mathbf{A} + \mathbf{B}_n)\,\mathbf{x}_n, \tag{1.1}$$

where $\mathbf{x}_n$ $(n = 0, 1, ...)$ are k-dimensional complex column vectors and $\mathbf{A}$ and $\mathbf{B}_n$ $(n = 0, 1, ...)$ are $k \times k$ matrices with complex entries.

Let $\|\cdot\|$ denote any norm of a vector or the associated induced norm of a square matrix.

Máté and Nevai [5] have proved that if the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ of $\mathbf{A}$ have distinct moduli and $\xi_1, \xi_2, ..., \xi_k$ are the corresponding eigenvectors, then

$$\lim_{n \to \infty} \|\mathbf{B}_n\| = 0 \tag{1.2}$$

implies that for every eventually nonzero solution $\{\mathbf{x}_n\}$ of (1.1) there exist an index $i \in \{1, ..., k\}$ and a sequence of complex numbers $\{\zeta_n\}$ such that

$$\lim_{n \to \infty} \zeta_n \mathbf{x}_n = \xi_i. \tag{1.3}$$

* Research supported in part by Hungarian National Foundation for Scientific Research Grant Nos. T 019846 and F 023772.

0021-9045/97 $25.00

In the case of the higher order scalar equation

$$u(n+1)=\sum_{i=0}^{k}(c_i+d_i(n))\,u(n-i), \tag{1.4}$$

where c_i $(0\leqslant i\leqslant k)$ are complex constants and $\{d_i(n)\}_{n=0}^{\infty}$ $(0\leqslant i\leqslant k)$ are complex sequences, the result of Máté and Nevai reduces to the classical theorem of Poincaré [8] which states that if the zeros $\lambda_0, \lambda_1, ..., \lambda_k$ of the characteristic equation

$$\lambda=\sum_{i=0}^{k}c_i\lambda^{-i} \tag{1.5}$$

of the corresponding equation with constants coefficients

$$u(n+1)=\sum_{i=0}^{k}c_iu(n-i) \tag{1.6}$$

have distinct moduli and

$$\lim_{n\to\infty}d_i(n)=0 \qquad (0\leqslant i\leqslant k), \tag{1.7}$$

then for every eventually nonzero solution $\{u(n)\}$ of (1.4) there exists an index $i\in\{0, 1, ..., k\}$ such that

$$\lim_{n\to\infty}\frac{u(n+1)}{u(n)}=\lambda_i. \tag{1.8}$$

Note that Poincaré's theorem has applications in the study of the asymptotic behavior of orthogonal polynomials (see [8, p. 252; 6, Section 2] for details).

Formula (1.3) does not give an asymptotic approximation of the solutions of (1.1), since the sequence $\{\zeta_n\}$ in (1.3) is not determined explicitly. Similarly, Poincaré's theorem asserts that the solutions of (1.4) have the property of convergence of ratios of successive values, but, for most purposes, we would like to have information about the solutions themselves. To obtain more precise asymptotic characterization of the solutions, it is necessary to replace (1.2) and (1.7) with stronger conditions. For Eq. (1.1) Coffman [2] and Li [4] have given results along these lines. For example, as a consequence of [4, Theorem I], we obtain that if the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ of $\mathbf{A}$ are nonzero and distinct, then

$$\sum_{n=0}^{\infty}\|\mathbf{B}_n\|<\infty \tag{1.9}$$

implies that for each $i \in \{1, ..., k\}$ Eq. (1.1) has a solution $\{\mathbf{x}_n\}$ such that

$$\lim_{n \to \infty} \lambda_i^{-n} \mathbf{x}_n = \xi_i, \tag{1.10}$$

where ξ_i is the eigenvector corresponding to λ_i. That is, (1.3) holds with $\zeta_n = \lambda_i^{-n}$ $(n = 0, 1, ...)$. In [10] Trench proved similar results under assumptions allowing conditional convergence of the series $\sum_{n=0}^{\infty} \mathbf{B}_n$. The corresponding result for the scalar equation (1.4) states (see, e.g., [2, Theorem 10.1]) that if the roots $\lambda_0, \lambda_2, ..., \lambda_k$ of (1.5) are nonzero and distinct, then

$$\sum_{n=0}^{\infty} |d_i(n)| < \infty \qquad (0 \leqslant i \leqslant k) \tag{1.11}$$

implies that for each $i \in \{0, 1, ..., k\}$ Eq. (1.4) has a solution $\{u(n)\}$ such that

$$\lim_{n \to \infty} \lambda_i^{-n} u(n) = 1. \tag{1.12}$$

In a recent paper [7] the author described the asymptotic behavior of the solutions of (1.4) under substantially weaker assumptions on the perturbation terms $\{d_i(n)\}$ than (1.11). The main result of [7] shows (see [7, Theorem 4.1]) that if the characteristic equation (1.5) has a dominant root λ_0 (i.e., λ_0 is a simple root and all other roots satisfy $|\lambda| < |\lambda_0|$) and the perturbations satisfy

$$\sum_{n=0}^{\infty} |d_i(n)|^2 < \infty \qquad (0 \leqslant i \leqslant k) \tag{1.13}$$

and

$$\sum_{n=0}^{\infty} |d_i(n+1) - d_i(n)| < \infty \qquad (0 \leqslant i \leqslant k), \tag{1.14}$$

then, for n_0 large enough, Eq. (1.4) has a solution $\{u(n)\}$ such that

$$\lim_{n \to \infty} \left[u(n) \middle/ \prod_{\nu = n_0}^{n-1} \left(\lambda_0 + \frac{1}{f'(\lambda_0)} \sum_{i=0}^{k} \lambda_0^{-i} d_i(\nu) \right) \right] = 1, \tag{1.15}$$

where f is the characteristic function defined by

$$f(\lambda) = \lambda - \sum_{i=0}^{k} c_i \lambda^{-i}.$$

However, the assumption that Eq. (1.5) has a dominant root seems to be too strict. In [7] it was conjectured that the same conclusion is true if in addition to (1.13) and (1.14) we assume that λ_0 is a simple root of (1.5) and all other roots satisfy $|\lambda| \neq |\lambda_0|$.

In this paper, among others, we confirm the above-mentioned conjecture. Actually, we prove a stronger result showing that assumption (1.14) can also be eliminated. Our main results are formulated in the following two theorems.

THEOREM 1. *Let λ_0 be a simple root of the characteristic function*

$$f(\lambda) = \lambda - \sum_{i=0}^{k} c_i \lambda^{-i}$$

for Eq. (1.6) *and suppose that if λ is any other root of f, then $|\lambda| \neq |\lambda_0|$. If*

$$\sum_{n=0}^{\infty} |d_i(n)|^2 < \infty \qquad (0 \leqslant i \leqslant k), \tag{1.13}$$

then, for sufficiently large n_0, (1.4) *has a solution $\{u(n)\}_{n=n_0}^{\infty}$ such that*

$$\lim_{n \to \infty} \left[u(n) \Big/ \prod_{\nu=n_0}^{n-1} \left(\lambda_0 + \frac{1}{f'(\lambda_0)} \sum_{i=0}^{k} \lambda_0^{-i} d_i(\nu) \right) \right] = 1. \tag{1.15}$$

Theorem 1 will be deduced from the following more general result concerning system (1.1).

THEOREM 2. *Suppose that λ_0 is a simple nonzero eigenvalue of* $\mathbf{A}$, *and that if λ is any other eigenvalue of* $\mathbf{A}$, *then $|\lambda| \neq |\lambda_0|$. Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be nonzero vectors such that $\mathbf{A}\boldsymbol{\xi} = \lambda_0 \boldsymbol{\xi}$ and $\mathbf{A}^* \boldsymbol{\eta} = \overline{\lambda_0} \boldsymbol{\eta}$.*[1] *If*

$$\sum_{n=0}^{\infty} \|\mathbf{B}_n\|^2 < \infty, \tag{1.16}$$

then, for sufficiently large n_0, (1.1) *has a solution $\{\mathbf{x}_n\}_{n=n_0}^{\infty}$ such that*

$$\lim_{n \to \infty} \left[\mathbf{x}_n \Big/ \prod_{\nu=n_0}^{n-1} (\lambda_0 + \beta_\nu) \right] = \boldsymbol{\xi}, \tag{1.17}$$

where $\{\beta_n\}_{n=0}^{\infty}$ is the sequence of complex numbers defined by

$$\beta_n = (\boldsymbol{\eta}^* \boldsymbol{\xi})^{-1} \boldsymbol{\eta}^* \mathbf{B}_n \boldsymbol{\xi}, \qquad n = 0, 1, \ldots. \tag{1.18}$$

[1] According to the standard notation, $\mathbf{A}^*$ and $\overline{\lambda_0}$ denote the conjugate transpose of $\mathbf{A}$ and the conjugate of λ_0, respectively.

Remark 1. Under the assumptions of Theorem 2, $\boldsymbol{\eta}^*\boldsymbol{\xi} \neq 0$. This can be shown by the following simple argument. Let $B = \{\boldsymbol{\xi}, \boldsymbol{\xi}_2, ..., \boldsymbol{\xi}_k\}$ be the basis for $\mathbb{C}^k$ consisting of generalized eigenvectors of $\mathbf{A}$. Since $\boldsymbol{\eta}$ is orthogonal to any generalized eigenvector of $\mathbf{A}$ corresponding to an eigenvalue different from λ_0 and λ_0 is a simple eigenvalue of $\mathbf{A}$, it follows that $\boldsymbol{\eta}^*\boldsymbol{\xi}_i = 0$, $2 \leqslant i \leqslant k$. Since $\boldsymbol{\eta} \neq \mathbf{0}$ can be written as a linear combination of $\boldsymbol{\xi}, \boldsymbol{\xi}_2, ..., \boldsymbol{\xi}_k$, this implies $\boldsymbol{\eta}^*\boldsymbol{\xi} \neq 0$. Thus, β_n $(n = 0, 1, ...)$ is well defined.

Remark 2. By a standard result on infinite products, if $\sum^\infty |a_\nu|^2 < \infty$, then $\sum^\infty a_\nu$ and $\prod^\infty (1 + a_\nu)$ converge or diverge together. Therefore, if in addition to the assumptions of Theorem 2, we assume that the series $\sum_{\nu=0}^\infty \beta_\nu$ converges (perhaps conditionally), then, for sufficiently large n_0, $\prod_{\nu=n_0}^\infty (1 + \beta_\nu/\lambda_0) = P$, where P is finite and nonzero. In this case Theorem 2 yields the existence of a solution $\{\mathbf{x}_n\}_{n=n_0}^\infty$ of (1.1) such that

$$\lim_{n \to \infty} \lambda_0^{-n} \mathbf{x}_n = \boldsymbol{\xi}.$$

The same argument applies to Theorem 1. Consequently, if in addition to the assumptions of Theorem 1 we assume that the series $\sum_{\nu=0}^\infty d_i(\nu)$ $(0 \leqslant i \leqslant k)$ all converge (perhaps conditionally), then, for sufficiently large n_0, (1.4) has a solution $\{u(n)\}_{n=n_0}^\infty$ such that

$$\lim_{n \to \infty} \lambda_0^{-n} u(n) = 1.$$

Remark 3. If the eigenvalues of $\mathbf{A}$ are nonzero and have distinct moduli, then Theorem 2 yields the existence of k linearly independent solutions (a fundamental system of solutions) of (1.1). A similar remark holds for the scalar equation (1.4).

The proof of Theorems 1 and 2 will be given in Section 3 after presenting some lemmas in Section 2.

2. LEMMAS

In this section, we establish some lemmas regarding l_2-sequences. We will use the following notation. Given a nonnegative integer n_1, denote $\mathbb{N}_{n_1} = \{n_1, n_1 + 1, ...\}$. Let $l_2(\mathbb{N}_{n_1})$ denote the set of those (complex) sequences $\omega = \{\omega_n\}_{n=n_1}^\infty$ for which $\sum_{n=n_1}^\infty |\omega_n|^2 < \infty$. With the norm

$$\|\omega\|_{l_2(\mathbb{N}_{n_1})} = \left(\sum_{n=n_1}^\infty |\omega_n|^2 \right)^{1/2}, \qquad \omega = \{\omega_n\}_{n=n_1}^\infty \in l_2(\mathbb{N}_{n_1}),$$

$l_2(\mathbb{N}_{n_1})$ is a Banach space.

LEMMA 1. *Let* $0<\mu<1$ *and let* $\gamma=\{\gamma_n\}_{n=n_1}^{\infty}\in l_2(\mathbb{N}_{n_1})$. *Define*

$$\varphi_n=\sum_{i=n_1}^{n-1}\mu^{n-1-i}\gamma_i, \qquad \psi_n=\sum_{i=n}^{\infty}\mu^{i-n+1}\gamma_i \qquad \text{for} \quad n\geqslant n_1. \tag{2.1}$$

Then the sequences $\varphi=\{\varphi_n\}_{n=n_1}^{\infty}$ *and* $\psi=\{\psi_n\}_{n=n_1}^{\infty}$ *belong to* $l_2(\mathbb{N}_{n_1})$ *and the estimates*

$$\|\varphi\|_{l_2(\mathbb{N}_{n_1})}\leqslant(1-\mu)^{-1}\,\|\gamma\|_{l_2(\mathbb{N}_{n_1})} \tag{2.2}$$

and

$$\|\psi\|_{l_2(\mathbb{N}_{n_1})}\leqslant\mu(1-\mu)^{-1}\,\|\gamma\|_{l_2(\mathbb{N}_{n_1})} \tag{2.3}$$

hold.

Proof. The sequences φ and ψ can be written in the form

$$\varphi=\sum_{m=1}^{\infty}\mu^{m-1}R^m\gamma$$

and

$$\psi=\sum_{m=0}^{\infty}\mu^{m+1}L^m\gamma,$$

where R, L: $l_2(\mathbb{N}_{n_1})\to l_2(\mathbb{N}_{n_1})$ are the right and left shift operators, that is, for $\gamma=\{\gamma_n\}_{n=n_1}^{\infty}$, $R\gamma=\{(R\gamma)_n\}_{n=n_1}^{\infty}$ and $L\gamma=\{(L\gamma)_n\}_{n=n_1}^{\infty}$ are defined by

$$(R\gamma)_n=\begin{cases}0 & \text{for} \quad n=n_1,\\ \gamma_{n-1} & \text{for} \quad n\geqslant n_1+1,\end{cases}$$

and

$$(L\gamma)_n=\gamma_{n+1} \qquad \text{for} \quad n\geqslant n_1.$$

Consequently,

$$\|\varphi\|_{l_2(\mathbb{N}_{n_1})}\leqslant\sum_{m=1}^{\infty}\mu^{m-1}\,\|R^m\|\,\|\gamma\|_{l_2(\mathbb{N}_{n_1})},$$

where $\|\cdot\|$ is the operator norm. Hence,

$$\|\varphi\|_{l_2(\mathbb{N}_{n_1})}\leqslant\sum_{m=1}^{\infty}\mu^{m-1}\,\|R\|^m\,\|\gamma\|_{l_2(\mathbb{N}_{n_1})}.$$

Similarly,

$$\|\psi\|_{l_2(\mathbb{N}_{n_1})} \leqslant \sum_{m=0}^{\infty} \mu^{m+1} \|L\|^m \|\gamma\|_{l_2(\mathbb{N}_{n_1})}.$$

Since $\|R\| = \|L\| = 1$, the last two estimates imply (2.2) and (2.3).

The following lemma will play an important role in the proof of Theorem 2.

LEMMA 2. *Let $0 < \mu < 1$ and let $\gamma = \{\gamma_n\}_{n=0}^{\infty} \in l_2(\mathbb{N}_0)$ be a sequence of positive numbers. Then for every $\eta > 0$ there exist an index n_1 and a sequence of positive numbers $\omega = \{\omega_n\}_{n=n_1}^{\infty} \in l_2(\mathbb{N}_{n_1})$ such that*

$$\sum_{i=n_1}^{n-1} \mu^{n-1-i} \gamma_i (1+\omega_i) + \sum_{i=n}^{\infty} \mu^{i-n+1} \gamma_i (1+\omega_i) = \eta \omega_n \tag{2.4}$$

for all $n \geqslant n_1$.

Proof. For $\omega = \{\omega_n\}_{n=n_1}^{\infty} \in l_2(\mathbb{N}_{n_1})$, define a sequence $T\omega = \{(T\omega)_n\}_{n=n_1}^{\infty}$ by

$$(T\omega)_n = \eta^{-1} \left(\sum_{i=n_1}^{n-1} \mu^{n-1-i} \gamma_i (1+\omega_i) + \sum_{i=n}^{\infty} \mu^{i-n+1} \gamma_i (1+\omega_i) \right)$$

for $n \geqslant n_1$. Since ω is a bounded sequence, $T\omega$ is well defined and, according to Lemma 1, $T\omega \in l_2(\mathbb{N}_{n_1})$. For any α, $\omega \in l_2(\mathbb{N}_{n_1})$ and $n \geqslant n_1$, we have

$$\begin{aligned} |(T\alpha - T\omega)_n| &= \eta^{-1} \left| \sum_{i=n_1}^{n-1} \mu^{n-1-i} \gamma_i (\alpha_i - \omega_i) + \sum_{i=n}^{\infty} \mu^{i-n+1} \gamma_i (\alpha_i - \omega_i) \right| \\ &\leqslant \eta^{-1} \max_{i \geqslant n_1} |\alpha_i - \omega_i| \, (\varphi_n + \psi_n) \end{aligned}$$

with φ_n and ψ_n given by (2.1). Consequently,

$$\begin{aligned} \|T\alpha - T\omega\|_{l_2(\mathbb{N}_{n_1})} &\leqslant \eta^{-1} \max_{i \geqslant n_1} |\alpha_i - \omega_i| \, (\|\varphi\|_{l_2(\mathbb{N}_{n_1})} + \|\psi\|_{l_2(\mathbb{N}_{n_1})}) \\ &\leqslant \eta^{-1} \max_{i \geqslant n_1} |\alpha_i - \omega_i| \, (1+\mu)(1-\mu)^{-1} \|\gamma\|_{l_2(\mathbb{N}_{n_1})}, \end{aligned}$$

where the last inequality is a consequence of (2.2) and (2.3). From this, in view of the inequality $\max_{i \geqslant n_1} |\alpha_i - \omega_i| \leqslant \|\alpha - \omega\|_{l_2(\mathbb{N}_{n_1})}$, we get

$$\|T\alpha - T\omega\|_{l_2(\mathbb{N}_{n_1})} \leqslant \eta^{-1} (1+\mu)(1-\mu)^{-1} \|\gamma\|_{l_2(\mathbb{N}_{n_1})} \|\alpha - \omega\|_{l_2(\mathbb{N}_{n_1})}. \tag{2.5}$$

Since $\|\gamma\|_{l_2(\mathbb{N}_{n_1})} \to 0$ as $n_1 \to \infty$, we can find n_1 so that

$$\|\gamma\|_{l_2(\mathbb{N}_{n_1})} < \eta(1+\mu)^{-1}(1-\mu).$$

Then $T\colon l_2(\mathbb{N}_{n_1}) \to l_2(\mathbb{N}_{n_1})$ is a contraction mapping (cf. (2.5)) and by Banach's theorem it has a unique fixed-point $\omega \in l_2(\mathbb{N}_{n_1})$. Evidently, ω satisfies (2.4). It remains to show that ω is a sequence of positive numbers. We know that ω can be written as a limit (in $l_2(\mathbb{N}_{n_1})$) of successive approximations

$$\omega^{[\nu+1]} = T\omega^{[\nu]}, \qquad \nu = 0, 1, 2, ...,$$

where $\omega^{[0]} \in l_2(\mathbb{N}_{n_1})$ is arbitrary. Taking $\omega^{[0]} \equiv 0$, we see by easy induction on ν that

$$\omega_n^{[\nu]} \geqslant \eta^{-1}\left(\sum_{i=n_1}^{n-1} \mu^{n-1-i}\gamma_i + \sum_{i=n}^{\infty} \mu^{i-n+1}\gamma_i\right) \tag{2.6}$$

for $n \geqslant n_1$ and $\nu = 1, 2, ...$ For any fixed $n \geqslant n_1$,

$$|\omega_n^{[\nu]} - \omega_n| \leqslant \|\omega^{[\nu]} - \omega\|_{l_2(\mathbb{N}_{n_1})} \to 0 \qquad \text{as} \quad \nu \to \infty.$$

Thus, $\omega_n = \lim_{\nu\to\infty} \omega_n^{[\nu]}$. Letting $\nu \to \infty$ in (2.6), we obtain

$$\omega_n \geqslant \eta^{-1}\left(\sum_{i=n_1}^{n-1} \mu^{n-1-i}\gamma_i + \sum_{i=n}^{\infty} \mu^{i-n+1}\gamma_i\right) > 0$$

for $n \geqslant n_1$ and the proof is complete.

3. PROOFS OF THE THEOREMS

Proof of Theorem 2. We shall prove Theorem 2 in two steps. First we give a proof in the case when $\lambda_0 = 1$ and then we show that the general case can be reduced to the previous one.

Step 1. Assume that $\lambda_0 = 1$. Let $\sigma(\mathbf{A})$ denote the spectrum of $\mathbf{A}$. Define

$$\sigma_{-1} = \{\lambda \in \sigma(\mathbf{A}) \colon |\lambda| < 1\},$$
$$\sigma_0 = \{1\},$$

and

$$\sigma_1 = \{\lambda \in \sigma(\mathbf{A}) \colon |\lambda| > 1\}.$$

Clearly, $\sigma(\mathbf{A}) = \sigma_{-1} \cup \sigma_0 \cup \sigma_1$ and we have the decomposition of $\mathbb{C}^k$ into a direct sum,

$$\mathbb{C}^k = X_{-1} \oplus X_0 \oplus X_1, \tag{3.1}$$

where X_i is the linear subspace generated by the generalized eigenvectors of $\mathbf{A}$ corresponding to those eigenvalues which belong to σ_i ($i = -1, 0, 1$). Let $L\colon \mathbb{C}^k \to \mathbb{C}^k$ be the linear operator associated with matrix $\mathbf{A}$, i.e.,

$$L(\mathbf{x}) = \mathbf{A}\mathbf{x} \qquad \text{for} \quad \mathbf{x} \in \mathbb{C}^k.$$

Denote

$$L_i = L|_{X_i},$$

the restriction of L to X_i ($i = -1, 0, 1$). It is well known that the subspaces X_i ($i = -1, 0, 1$) are invariant under transformation L. Thus, L_i maps X_i into itself ($i = -1, 0, 1$). Let $\sigma(L_i)$ and $\varrho(L_i)$ denote the spectrum and the spectral radius of L_i, respectively. Obviously,

$$\sigma(L_i) = \sigma_i \qquad (i = -1, 0, 1) \tag{3.2}$$

and

$$\varrho(L_i) = \max\{|\lambda| \colon \lambda \in \sigma_i\} \qquad (i = -1, 0, 1). \tag{3.3}$$

For the spectral radius we also have the formula [3, Section 90, Excercise 5(e)]

$$\varrho(L_i) = \lim_{n \to \infty} \sqrt[n]{\|L_i^n\|} \qquad (i = -1, 0, 1), \tag{3.4}$$

where L_i^n denotes the nth iteration of L and $\|\cdot\|$ is the operator norm. The decomposition (3.1) of $\mathbb{C}^k$ defines three projection operators $\pi_i\colon \mathbb{C}^k \to X_i$ ($i = -1, 0, 1$) in the following way: if $\mathbf{x} \in \mathbb{C}^k$ is written as $\mathbf{x} = \mathbf{x}_{-1} + \mathbf{x}_0 + \mathbf{x}_1$, where $\mathbf{x}_i \in X_i$ ($i = -1, 0, 1$), then $\pi_i(\mathbf{x}) \overset{\text{def}}{=} \mathbf{x}_i$ ($i = -1, 0, 1$). These projections have the properties

$$\pi_i \circ \pi_i = \pi_i \qquad (i = -1, 0, 1) \tag{3.5}$$

$$\pi_i \circ \pi_j = 0 \qquad \text{if} \quad i \neq j, \quad i, j \in \{-1, 0, 1\}, \tag{3.6}$$

and

$$\pi_{-1} + \pi_0 + \pi_1 = \mathrm{id}_{\mathbb{C}^k}. \tag{3.7}$$

Since the subspaces X_i are invariant under transformation L, for each $i=-1, 0, 1$, π_i commutes with L (cf. [3, Section 43, Theorem 2]); i.e.,

$$L \circ \pi_i = \pi_i \circ L \qquad (i=-1, 0, 1). \tag{3.8}$$

Since $\lambda_0 = 1$ is a simple eigenvalue of $\mathbf{A}$, $X_0 = \operatorname{span}\{\boldsymbol{\xi}\}$ and, hence,

$$L \circ \pi_0 = \pi_0. \tag{3.9}$$

From (3.3) and the definition of σ_{-1}, it follows that $\varrho(L_{-1}) < 1$. Further, since $0 \notin \sigma_1 = \sigma(L_1)$, L_1 is invertible. If L_1^{-1} denotes the inverse of L_1, then $\sigma(L_1^{-1}) = \{\lambda^{-1} : \lambda \in \sigma_1\}$ and, hence, $\varrho(L_1^{-1}) < 1$. From this and the formula for the spectral radius (cf. (3.4)), it follows that if μ is a constant satisfying

$$\max\{\varrho(L_{-1}), \varrho(L_1^{-1})\} < \mu < 1,$$

then there exists a constant $K > 0$ such that

$$\|L_{-1}^n\| \leqslant K\mu^n \tag{3.10}$$

and

$$\|L_1^{-n}\| \leqslant K\mu^n \tag{3.11}$$

for all $n \geqslant 0$. (Here $L_1^{-n} = (L_1^{-1})^n$ for $n \geqslant 0$.)

Let $\{\tilde{\beta}_n\}_{n=0}^{\infty}$ be the sequence of complex numbers defined by

$$\pi_0(\mathbf{B}_n \boldsymbol{\xi}) = \tilde{\beta}_n \boldsymbol{\xi} \qquad \text{for} \quad n \geqslant 0. \tag{3.12}$$

Since $\boldsymbol{\eta}$ is orthogonal to any generalized eigenvector of $\mathbf{A}$ corresponding to an eigenvalue different from $\lambda_0 = 1$, we have (cf. (3.7))

$$\boldsymbol{\eta}^* \mathbf{x} = \boldsymbol{\eta}^* \pi_0(\mathbf{x}) \qquad \text{for all} \quad \mathbf{x} \in \mathbb{C}^k \tag{3.13}$$

which, together with (3.12), implies

$$\boldsymbol{\eta}^* \mathbf{B}_n \boldsymbol{\xi} = \boldsymbol{\eta}^* \pi_0(\mathbf{B}_n \boldsymbol{\xi}) = \tilde{\beta}_n \boldsymbol{\eta}^* \boldsymbol{\xi}. \tag{3.14}$$

Consequently (cf. Remark 1)

$$\tilde{\beta}_n = \beta_n \qquad \text{for} \quad n \geqslant 0, \tag{3.15}$$

where β_n is given by (1.18).

From (3.12) and (3.15), we see that

$$|\beta_n| \leqslant \|\pi_0\| \, \|\mathbf{B}_n\| \qquad \text{for} \quad n \geqslant 0. \tag{3.16}$$

Therefore (by (1.16))

$$\beta = \{\beta_n\}_{n=0}^{\infty} \in l_2(\mathbb{N}_0). \tag{3.17}$$

Specially, $\beta_n \to 0$ as $n \to \infty$ and, hence,

$$1 + \beta_n \neq 0 \qquad \text{for} \quad n \geqslant n_0 \tag{3.18}$$

whenever n_0 is large enough.

Choose n_0 so large that (3.18) is fulfilled and introduce the transformation

$$\mathbf{y}_n = \frac{\mathbf{x}_n}{\prod_{i=n_0}^{n-1} (1+\beta_i)} \qquad \text{for} \quad n \geqslant n_0. \tag{3.19}$$

A sequence $\{\mathbf{x}_n\}_{n=n_0}^{\infty}$ is a solution of (1.1) if and only if $\{\mathbf{y}_n\}_{n=n_0}^{\infty}$ satisfies

$$\mathbf{y}_{n+1} = \mathbf{A}\mathbf{y}_n + (\mathbf{B}_n - \mathbf{C}_n)\,\mathbf{y}_n \tag{3.20}$$

for $n \geqslant n_0$, where

$$\mathbf{C}_n = \frac{\beta_n}{1+\beta_n}\,(\mathbf{A} + \mathbf{B}_n). \tag{3.21}$$

Consequently, it suffices to show that (3.20) has a solution $\{\mathbf{y}_n\}_{n=n_0}^{\infty}$ such that $\mathbf{y}_n \to \xi$ as $n \to \infty$. We have

$$\pi_0(\mathbf{C}_n\xi) = \frac{\beta_n}{1+\beta_n}\,(\pi_0(\mathbf{A}\xi) + \pi_0(\mathbf{B}_n\xi)) = \frac{\beta_n}{1+\beta_n}\,\xi + \frac{\beta_n}{1+\beta_n}\,\pi_0(\mathbf{B}_n\xi) = \pi_0(\mathbf{B}_n\xi),$$

the last equality being a consequence of (3.12) and (3.15). Consequently

$$\pi_0(\mathbf{C}_n\pi_0(\mathbf{x})) = \pi_0(\mathbf{B}_n\pi_0(\mathbf{x})) \tag{3.22}$$

for all $\mathbf{x} \in \mathbb{C}^k$. Define

$$\gamma_n = \|\mathbf{B}_n\| + \|\mathbf{C}_n\| \qquad \text{for} \quad n \geqslant 0. \tag{3.23}$$

From (1.16) and (3.17), it follows that

$$\|(1+\beta_n)^{-1}\,(\mathbf{A} + \mathbf{B}_n)\| \to \|\mathbf{A}\| \qquad \text{as} \quad n \to \infty$$

and, hence, (cf. (3.21)) $\|\mathbf{C}_n\| \leqslant \text{const}\,|\beta_n|$. Therefore (by (1.16) and (3.17))

$$\gamma = \{\gamma_n\}_{n=0}^{\infty} \in l_2(\mathbb{N}_0). \tag{3.24}$$

Let B denote the linear space of those vector sequences $y=\{\mathbf{y}_n\}_{n=n_0}^{\infty}$, $\mathbf{y}_n\in\mathbb{C}^k$ for $n\geqslant n_0$, for which

$$\|y\|_B \overset{\text{def}}{=} \sup_{n\geqslant n_0}\|\pi_0(\mathbf{y}_n)\| + \sup_{n\geqslant n_0}\omega_n^{-1}\,\|(\pi_{-1}+\pi_1)(\mathbf{y}_n)\|<\infty, \qquad (3.25)$$

where $\omega=\{\omega_n\}_{n=n_0}^{\infty}\in l_2(\mathbb{N}_{n_0})$ is a sequence of positive numbers which will be specified later. It is easy to show that $\|\cdot\|_B$ is a norm on B and $(B, \|\cdot\|_B)$ is a Banach space (cf. [1, Lemma 3]).

For $y=\{\mathbf{y}_n\}_{n=n_0}^{\infty}\in B$, define a vector sequence $Fy=\{\mathbf{F}\mathbf{y}_n\}_{n=n_0}^{\infty}$ by

$$\mathbf{F}\mathbf{y}_n=\boldsymbol{\xi}+\sum_{i=n_0}^{n-1} L_{-1}^{n-1-i}(\pi_{-1}((\mathbf{B}_i-\mathbf{C}_i)\,\mathbf{y}_i))$$
$$-\sum_{i=n}^{\infty}\pi_0((\mathbf{B}_i-\mathbf{C}_i)\,\mathbf{y}_i)-\sum_{i=n}^{\infty}L_1^{n-1-i}(\pi_1((\mathbf{B}_i-\mathbf{C}_i)\,\mathbf{y}_i))$$

for $n\geqslant n_0$. By virtue of (3.7) and (3.22), we have

$$\pi_0((\mathbf{B}_i-\mathbf{C}_i)\,\mathbf{y}_i)=\pi_0((\mathbf{B}_i-\mathbf{C}_i)((\pi_{-1}+\pi_1)(\mathbf{y}_i)+\pi_0(\mathbf{y}_i)))$$
$$=\pi_0((\mathbf{B}_i-\mathbf{C}_i)(\pi_{-1}+\pi_1)(\mathbf{y}_i)).$$

Consequently, $\mathbf{F}\mathbf{y}_n$ can be written in the form

$$\mathbf{F}\mathbf{y}_n=\boldsymbol{\xi}+\sum_{i=n_0}^{n-1} L_{-1}^{n-1-i}(\pi_{-1}((\mathbf{B}_i-\mathbf{C}_i)\,\mathbf{y}_i))$$
$$-\sum_{i=n}^{\infty}\pi_0((\mathbf{B}_i-\mathbf{C}_i)(\pi_{-1}+\pi_1)(\mathbf{y}_i))-\sum_{i=n}^{\infty}L_1^{n-1-i}(\pi_1((\mathbf{B}_i-\mathbf{C}_i)\,\mathbf{y}_i))$$

for $n\geqslant n_0$. From this, by virtue of (3.5), (3.6), and (3.8), we get

$$\pi_0(\mathbf{F}\mathbf{y}_n)=\boldsymbol{\xi}-\sum_{i=n}^{\infty}\pi_0((\mathbf{B}_i-\mathbf{C}_i)(\pi_{-1}+\pi_1)(\mathbf{y}_i)), \qquad (3.26)$$

$$\pi_{-1}(\mathbf{F}\mathbf{y}_n)=\sum_{i=n_0}^{n-1}L_{-1}^{n-1-i}(\pi_{-1}((\mathbf{B}_i-\mathbf{C}_i)\,\mathbf{y}_i)), \qquad (3.27)$$

$$\pi_1(\mathbf{F}\mathbf{y}_n)=-\sum_{i=n}^{\infty}L_1^{n-1-i}(\pi_1((\mathbf{B}_i-\mathbf{C}_i)\,\mathbf{y}_i)) \qquad (3.28)$$

for $n\geqslant n_0$.

From the definition of the norm on B (cf. (3.25)), it follows that

$$\|\pi_0(\mathbf{y}_n)\| \leqslant \|y\|_B, \qquad \|(\pi_{-1}+\pi_1)(\mathbf{y}_n)\| \leqslant \omega_n \|y\|_B \tag{3.29}$$

for $y=\{\mathbf{y}_n\}_{n=n_0}^{\infty} \in B$ and $n \geqslant n_0$. Obviously,

$$\|\mathbf{y}_n\| = \|\pi_0(\mathbf{y}_n)+(\pi_{-1}+\pi_1)(\mathbf{y}_n)\| \leqslant \|\pi_0(\mathbf{y}_n)\| + \|(\pi_{-1}+\pi_1)(\mathbf{y}_n)\|$$

which, in view of (3.29), gives

$$\|\mathbf{y}_n\| \leqslant (1+\omega_n) \|y\|_B \tag{3.30}$$

for $y=\{\mathbf{y}_n\}_{n=n_0}^{\infty} \in B$ and $n \geqslant n_0$.

By virtue of (3.23), (3.26), and (3.29), we can estimate the norm of $\pi_0(\mathbf{F}\mathbf{y}_n)$ as

$$\begin{aligned}\|\pi_0(\mathbf{F}\mathbf{y}_n)\| &\leqslant \|\xi\| + \sum_{i=n}^{\infty} \|\pi_0\| \, \|\mathbf{B}_i - \mathbf{C}_i\| \, \|(\pi_{-1}+\pi_1)(\mathbf{y}_i)\| \\ &\leqslant \|\xi\| + \sum_{i=n}^{\infty} \|\pi_0\| \, \gamma_i \omega_i \|y\|_B \\ &\leqslant \|\xi\| + \|\pi_0\| \, \|y\|_B \, \|\gamma\|_{l_2(\mathbb{N}_{n_0})} \, \|\omega\|_{l_2(\mathbb{N}_{n_0})}\end{aligned}$$

for $n \geqslant n_0$, where the last inequality is a consequence of the Schwarz inequality. Thus,

$$\sup_{n \geqslant n_0} \|\pi_0(\mathbf{F}\mathbf{y}_n)\| < \infty. \tag{3.31}$$

Taking into account (3.10) and (3.11), we obtain (cf. (3.30))

$$\begin{aligned}\|(\pi_{-1}+\pi_1)(\mathbf{F}\mathbf{y}_n)\| &\leqslant \sum_{i=n_0}^{n-1} \|L_{-1}^{n-1-i}\| \, \|\pi_{-1}\| \, \|\mathbf{B}_i - \mathbf{C}_i\| \, \|\mathbf{y}_i\| \\ &\quad + \sum_{i=n}^{\infty} \|L_1^{n-1-i}\| \, \|\pi_1\| \, \|\mathbf{B}_i - \mathbf{C}_i\| \, \|\mathbf{y}_i\| \\ &\leqslant \sum_{i=n_0}^{n-1} K\mu^{n-1-i} \|\pi_{-1}\| \, \gamma_i (1+\omega_i) \|y\|_B \\ &\quad + \sum_{i=n}^{\infty} K\mu^{i-n+1} \|\pi_1\| \, \gamma_i (1+\omega_i) \|y\|_B.\end{aligned}$$

Hence,

$$\|(\pi_{-1}+\pi_1)(\mathbf{Fy}_n)\| \leqslant K(\|\pi_{-1}\|+\|\pi_1\|)\left(\sum_{i=n_0}^{n-1} \mu^{n-1-i}\gamma_i(1+\omega_i) + \sum_{i=n}^{\infty} \mu^{i-n+1}\gamma_i(1+\omega_i)\right)\|y\|_B \tag{3.32}$$

for $n \geqslant n_0$.

By Lemma 2, there exist an index n_1 and a sequence of positive numbers $\omega = \{\omega_n\}_{n=n_1}^{\infty} \in l_2(\mathbb{N}_{n_1})$ such that

$$\sum_{i=n_1}^{n-1} \mu^{n-i-1}\gamma_i(1+\omega_i) + \sum_{i=n}^{\infty} \mu^{i-n+1}\gamma_i(1+\omega_i) = \tfrac{1}{3}(K(\|\pi_{-1}\|+\|\pi_1\|))^{-1}\,\omega_n$$

for $n \geqslant n_1$. Consequently, if $n_0 \geqslant n_1$ and ω is chosen as above, then (cf. (3.32))

$$\|(\pi_{-1}+\pi_1)(\mathbf{Fy}_n)\| \leqslant \tfrac{1}{3}\omega_n\,\|y\|_B \qquad \text{for} \quad n \geqslant n_0 \tag{3.33}$$

and, hence,

$$\sup_{n \geqslant n_0} \omega_n^{-1}\,\|(\pi_{-1}+\pi_1)(\mathbf{Fy}_n)\| \leqslant \tfrac{1}{3}\,\|y\|_B. \tag{3.34}$$

From (3.31) and (3.34), we see that if $n_0 \geqslant n_1$ and ω is chosen as before, then Fy is well defined and $Fy \in B$.

Let $y, z \in B$. For $n \geqslant n_0$, we have (cf. (3.26))

$$\begin{aligned}\|\pi_0(\mathbf{Fy}_n - \mathbf{Fz}_n)\| &= \left\|\sum_{i=n}^{\infty} \pi_0((\mathbf{B}_i - \mathbf{C}_i)(\pi_{-1}+\pi_1)(\mathbf{y}_i - \mathbf{z}_i))\right\| \\ &\leqslant \sum_{i=n}^{\infty} \|\pi_0\|\;\|\mathbf{B}_i - \mathbf{C}_i\|\;\|(\pi_{-1}+\pi_1)(\mathbf{y}_i - \mathbf{z}_i)\| \\ &\leqslant \sum_{i=n}^{\infty} \|\pi_0\|\;\gamma_i\omega_i\;\|y-z\|_B.\end{aligned}$$

Hence

$$\|\pi_0(\mathbf{Fy}_n - \mathbf{Fz}_n)\| \leqslant \|\pi_0\|\;\|\gamma\|_{l_2(\mathbb{N}_{n_0})}\;\|\omega\|_{l_2(\mathbb{N}_{n_1})}\;\|y-z\|_B, \tag{3.35}$$

provided $n \geqslant n_0 \geqslant n_1$.

By similar estimates as in the proof of (3.33), we obtain

$$\|(\pi_{-1}+\pi_1)(\mathbf{F}\mathbf{y}_n-\mathbf{F}\mathbf{z}_n)\| \leqslant \tfrac{1}{3}\,\omega_n\,\|y-z\|_B, \tag{3.36}$$

provided $n \geqslant n_0 \geqslant n_1$.

Choose $n_0 \geqslant n_1$ so large that (3.18) is fulfilled; moreover,

$$\|\gamma\|_{l_2(\mathbb{N}_{n_0})} \leqslant \tfrac{1}{3}(\|\pi_0\|\ \|\omega\|_{l_2(\mathbb{N}_{n_1})})^{-1}.$$

(Such an index certainly exists, since $\|\gamma\|_{l_2(\mathbb{N}_{n_0})} \to 0$ as $n_0 \to \infty$.) Then estimates (3.35) and (3.36) imply that

$$\|Fy-Fz\|_B \leqslant \tfrac{2}{3}\,\|y-z\|_B$$

for all $y, z \in B$. Thus, $F\colon B \to B$ is a contraction mapping and by Banach's fixed-point theorem there exists a unique $y \in B$ such that $y = Fy$. From (3.7)–(3.9), it follows easily that this fixed-point $y = \{\mathbf{y}_n\}_{n=n_0}^{\infty}$ is a solution of (3.20); moreover, $\mathbf{y}_n \to \xi$ as $n \to \infty$. In view of relation (3.19) between the solutions of (1.1) and (3.20), this completes the proof in the case when $\lambda_0 = 1$.

Step 2. Assume that λ_0 is an arbitrary simple nonzero eigenvalue of $\mathbf{A}$. Consider the transformation

$$\mathbf{z}_n = \mathbf{x}_n \lambda_0^{-(n-n_0)} \qquad \text{for} \quad n \geqslant n_0.$$

A sequence $\{\mathbf{x}_n\}_{n=n_0}^{\infty}$ is a solution of (1.1) if and only if $\{\mathbf{z}_n\}_{n=n_0}^{\infty}$ is a solution of the equation

$$\mathbf{z}_{n+1} = (\tilde{\mathbf{A}} + \tilde{\mathbf{B}}_n)\,\mathbf{z}_n, \tag{3.37}$$

where

$$\tilde{\mathbf{A}} = \lambda_0^{-1}\mathbf{A}, \qquad \tilde{\mathbf{B}}_n = \lambda_0^{-1}\mathbf{B}_n$$

for $n \geqslant 0$. Obviously, $\tilde{\lambda}_0 = 1$ is a simple eigenvalue of both $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^*$ with the same eigenvectors ξ and $\boldsymbol{\eta}$, respectively. According to the previous part of the proof (Step 1), if n_0 is sufficiently large, then (3.37) has a solution $\{\mathbf{z}_n\}_{n=n_0}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{\mathbf{z}_n}{\prod_{i=n_0}^{n-1}(1+\tilde{\beta}_i)} = \xi, \tag{3.38}$$

where

$$\tilde{\beta}_n = (\boldsymbol{\eta}^*\xi)^{-1}\,\boldsymbol{\eta}^*\tilde{\mathbf{B}}_n\xi = \lambda_0^{-1}\beta_n$$

with β_n given by (1.18). But,

$$\frac{\mathbf{z}_n}{\prod_{i=n_0}^{n-1}(1+\tilde{\beta}_i)} = \frac{\mathbf{x}_n \lambda_0^{-(n-n_0)}}{\prod_{i=n_0}^{n-1}(1+\lambda_0^{-1}\beta_i)} = \frac{\mathbf{x}_n}{\prod_{i=n_0}^{n-1}(\lambda_0+\beta_i)}$$

for $n \geqslant n_0$. Consequently, (3.38) reduces to (1.17) and the proof is complete.

Proof of Theorem 1. We shall prove Theorem 1 by applying Theorem 2. Define the column vector $\mathbf{x}_n = \mathrm{col}(x_{0n}, x_{1n}, ..., x_{kn})$ for $n \geqslant 0$ by putting

$$x_{in} = u(n-i) \qquad (0 \leqslant i \leqslant k).$$

The recurrence equations

$$x_{0,\, n+1} = \sum_{j=0}^{k} (c_j + d_j(n))\, x_{jn},$$

$$x_{i,\, n+1} = x_{i-1,\, n} \qquad (1 \leqslant i \leqslant k)$$

for $n \geqslant 0$ are clearly equivalent to Eq. (1.4). These recurrence equations can be written in the form as given in (1.1). The $(k+1) \times (k+1)$ matrix $\mathbf{A}$ has the form $\mathbf{A} = [a_{ij}]_{0 \leqslant i, j \leqslant k}$, where $a_{0j} = c_j$ $(0 \leqslant j \leqslant k)$, $a_{i,\, i-1} = 1$ $(1 \leqslant i \leqslant k)$ and all other a_{ij} are 0; for $n \geqslant 0$ the matrix $\mathbf{B}_n$ has the form $\mathbf{B}_n = [b_{ij}(n)]_{0 \leqslant i, j \leqslant k}$, where $b_{0j}(n) = d_j(n)$ $(0 \leqslant j \leqslant k)$ and all other $b_{ij}(n)$ are 0. That is,

$$\mathbf{A} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{k-1} & c_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$\mathbf{B}_n = \begin{pmatrix} d_0(n) & d_1(n) & \cdots & d_{k-1}(n) & d_k(n) \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Clearly,

$$\det(\mathbf{A}-\lambda\mathbf{I})=(-1)^k\left(\sum_{i=0}^{k} c_i\lambda^{k-i}-\lambda^{k+1}\right).$$

Consequently, λ_0 from Theorem 1 is a simple nonzero eigenvalue of $\mathbf{A}$ and all other eigenvalues satisfy $|\lambda|\neq|\lambda_0|$. The eigenvector $\boldsymbol{\xi}$ of $\mathbf{A}$ corresponding to λ_0 has the form

$$\boldsymbol{\xi}=\operatorname{col}(1, \lambda_0^{-1}, \lambda_0^{-2}, ..., \lambda_0^{-k}).$$

An easy computation shows that

$$\boldsymbol{\eta}=\operatorname{col}(1, \overline{\lambda_0}-\overline{c_0}, \overline{\lambda_0}^2-\overline{c_0}\,\overline{\lambda_0}-\overline{c_1}, ..., \overline{\lambda_0}^k-\overline{c_0}\,\overline{\lambda_0}^{k-1}$$
$$-\overline{c_1}\,\overline{\lambda_0}^{k-2}-\cdots-\overline{c_{k-2}}\,\overline{\lambda_0}-\overline{c_{k-1}})$$

is an eigenvector of the adjoint matrix $\mathbf{A}^*$ corresponding to the eigenvalue $\overline{\lambda_0}$. Hence,

$$\boldsymbol{\eta}^*\boldsymbol{\xi}=k+1-\sum_{i=0}^{k-1}(k-i)\,c_i\lambda_0^{-i-1}. \tag{3.39}$$

We have

$$(\lambda^k f(\lambda))'=\lambda^k f'(\lambda)+k\lambda^{k-1}f(\lambda). \tag{3.40}$$

On the other hand,

$$(\lambda^k f(\lambda))'=\left(\lambda^{k+1}-\sum_{i=0}^{k} c_i\lambda^{k-i}\right)'=(k+1)\,\lambda^k-\sum_{i=0}^{k-1}(k-i)\,c_i\lambda^{k-i-1}. \tag{3.41}$$

Comparing the right-hand sides of (3.40) and (3.41) and taking into account that $f(\lambda_0)=0$, we get

$$f'(\lambda_0)=k+1-\sum_{i=0}^{k-1}(k-i)\,c_i\lambda_0^{-i-1}=\boldsymbol{\eta}^*\boldsymbol{\xi} \qquad \text{(cf. (3.39))}.$$

Finally,

$$\boldsymbol{\eta}^*\mathbf{B}_n\boldsymbol{\xi}=\sum_{i=0}^{k}\lambda_0^{-i}d_i(n)$$

for $n \geqslant 0$. Consequently,

$$\beta_n = (\boldsymbol{\eta}^* \xi)^{-1} \boldsymbol{\eta}^* \mathbf{B}_n \xi = \frac{1}{f'(\lambda_0)} \sum_{i=0}^{k} \lambda_0^{-i} d_i(n)$$

and the assertion follows from conclusion (1.17) of Theorem 2.

ACKNOWLEDGMENT

The author is grateful to the referee for his observation formulated in Remark 2 and his suggestions to improve the presentation of this paper.

REFERENCES

1. R. P. Agarwal and M. Pituk, Convergence to equilibria in recurrence equations, *Comput. Math. Appl.*, to appear.
2. C. V. Coffman, Asymptotic behavior of solutions of ordinary difference equations, *Trans. Amer. Math. Soc.* **110** (1964), 22–51.
3. P. R. Halmos, "Finite-Dimensional Vector Spaces," Springer-Verlag, New York, 1974.
4. Z. Li, The asymptotic estimates of solutions of difference equations, *J. Math. Anal. Appl.* **94** (1983), 181–192.
5. A. Máté and P. Nevai, A generalization of Poincaré's theorem for recurrence equations, *J. Approx. Theory* **63** (1990), 92–97.
6. A. Máté, P. Nevai, and W. Van Assche, The support of measures associated with orthogonal polynomials and the spectra of the related self-adjoint operators, manuscript, 1988.
7. M. Pituk, Asymptotic behavior of a Poincaré difference equation, *J. Differ. Equations Appl.* **3** (1997), 33–53.
8. H. Poincaré, Sur les équations linéaires aux différentielles et aux différences finies, *Amer. J. Math.* **7** (1885), 203–258.
9. W. F. Trench, Asymptotic behavior of solutions of Poincaré difference equations, *Proc. Amer. Math. Soc.* **119** (1993), 431–438.
10. W. F. Trench, Asymptotic behavior of solutions of Poincaré recurrence systems, *Comput. Math. Appl.* **28** (1994), 317–324.